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Symmetrysets*

DAVID J. WINTER

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

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Combinatorial structures R are introduced which, in the presence of structure preserving symmetries at some or all points, determine a system or roots $\widehat{S(R)}$ in the sense of Bourbaki with 0 added.

INTRODUCTION

In this paper simple combinatorial structures called *productsets* and *relationsets* R are discussed which, in the presence of structure preserving *symmetries* at some or all elements, have associated systems of roots $\widehat{S(R)}$ in the sense of Bourbaki [1] with 0 added.

Those R isomorphic to systems of roots with 0 added are characterized by certain *rootsystem* properties. These rootsystem properties are easily verified for the set R of roots of a classical Lie algebra L, using certain key properties of L derived in Seligman [5] for $p > 7$. They are also easily verified for reduced symmetrysets contained in abelian groups with no 2, 3, 5, 7 torsion.

1. REFLECTIONS AND SYMMETRIES

Let G be a group with identity 1 and let R be a subset of G containing 1. Then R has a partial product $\pi(a, b) = ab$ defined on $D = \{(a, b) \in R \times R \mid ab \in R\}$, and R *together with* π *is a productset in the sense of the following definition.*

1.1 DEFINITION. A *productset* is a set R together with a function π froma subset D of $R \times R$ to R, denoted $\pi(a, b) = ab$ and called the *partial product* of R, such that:

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(1) R has an identity element 1 such that $(1, a)$, $(a, 1) \in D$ and $1a = a1 = a$ for all $a \in R$;

(2) $ab = ac$ implies $b = c$ for all (a, b), $(a, c) \in D$.

If $(a, b) \in D$ implies $(b, a) \in D$ and $ab = ba$ for all $a, b \in R$, R is *abelian*.

Throughout this paper, R denotes a finite productset with identity 1, partial product π and the domain D of π. The motivating example is that of a subset R of a group G, such as the set R of roots of a Lie algebra.

We define $a^i b$ for a, $b \in R$, $i \in \mathbb{Z}$ as follows:

(1) $a^{\circ} b = b$;

(2) $a^i b = a(a^{i-1} b)$ for all $i \geqslant 1$ for which $a^{i-1} b$ and $a(a^{i-1} b)$ are defined;

(3) $a^{-i} b = d$ if $b = a^i d$ $(i \geqslant 0)$.

We let $a^i b \in R$ indicate that $a^i b$ is defined, and we let $a^i b \notin R$ indicate that $a^i b$ is not defined (an abuse of language). Note that $a^i(a^j b) = a^{i+j} b$ for all i, $j \in \mathbb{Z}$ for which $a^j b$, $a^i(a^j b) \in R$.

Homomorphisms of productsets R_1, R_2 are mappings $f\colon R_1 \to R_2$ such that $f(ab) = f(a) f(b)$ for all a, b, $ab \in R_1$. A homomorphism of productsets $f\colon R_1 \to R_2$ is an *isomorphism* (*automorphism* if $R_1 = R_2$) if f is bijective and both f and f^{-1} are homomorphisms. The set of homomorphisms from R_1 to R_2 is denoted $\operatorname{Hom}(R_1, R_2)$, and $\operatorname{Aut} R$ denotes the group of automorphisms of R. Note that $f(a^i b) = f(a)^i\, b$ for $f \in \operatorname{Hom}(R_1, R_2)$, $i \in \mathbb{Z}$, a, b, $a^i b \in R_1$.

For $a \in R$ and $S \subset R$, the relation $\{(x, y) \in S \mid y = ax\}$ generates an equivalence relation on S. The corresponding equivalence class of $b \in S$ is the *string* $S_b(a) = \{a^{-r} b, \ldots, b, \ldots, a^q b\}$. If $a^{-(r+1} b$, $a_{q+1} b \notin S$, we say that the string $S_b(a)$ is *bounded* of *length* $q + r$.

If all strings $S_b(a)$ $(b \in S)$ are bounded, we introduce the *reflection* $r_a\colon S \to S$, which is the bijection from S to itself reversing each string $S_b(a) = \{a^{-r} b, \ldots, a^q b\}$:

$$r_a(a^i b) = a^{q-r-i} b.$$

In particular, $r_a(b) = a^{-a^*(b)} b$, where $a^*(b) = r - q$, the *Cartan integer* of b at a. Clearly, r_a is a symmetry of S at a in the following sense.

1.2 Definition. A *symmetry* of S at a is a bijection $s\colon S \to S$ such that

(1) $s\, S_b(a) = S_b(a)$ for all $b \in S$;

(2) $s(a^i b) = a^{-i} s(b)$ for all $b \in S$ and all $a^i b \in S_b(a)$ $(-r \leqslant i \leqslant q)$.

Clearly, a symmetry s of S at a has period 2. Moreover, if all strings $S_b(a)$ $(b \in S)$ are bounded, r_a is the only symmetry of S at a.

2. Reflectionsets and Symmetrysets

A *reflectionset* is a finite productset R such that

(1) the strings $R_b(a)$ $(a, b \in R, a \neq 1)$ are all bounded;

(2) the reflections $r_a(a \in R, a \neq 1)$ are automorphisms of R.

The subgroup $W(R)$ of Aut R generated by the reflections $r_a(a \in R, a \neq 1)$ is called the *Weyl group* of R.

Clearly, a reflectionset is a symmetryset in the following sense.

2.1 Definition. A *symmetryset* is a finite productset R such that Aut R contains a symmetry s_a of R at a for all $a \in R$, $a \neq 1$.

In a symmetryset R, each element $a \in R$ has a unique *inverse* $b \in R$ such that $ab = ba = 1$. This follows from the equations $s_a(a) = s_a(a1) = a^{-1}s_a(1) = a^{-1}1$, $1 = a(a^{-1}1)$, $1 = s_a(a(a^{-1}1)) = s_a(a)\, s_a(a^{-1}1) = (a^{-1}1)\,(as_a(1)) = (a^{-1}1)a$. Clearly, the inverse of a is $a^{-1}1$, which we denote henceforth by a^{-1}. Clearly, $s_a(a) = a^{-1}$.

If s_a is the reflection $r_a(b) = a^{-a^*(b)}b$, we have $a^*(a) = 2$. For example, $R_a(a) = \{a^{-r}a,..., a^q a\} = \{a^{1-r}1,..., a^{-1}1, 1, a1,..., a^{q+1}1\}$ and $r_a(1) = 1$ implies that $-(1-r) = q+1$ and $a^*(a) = r - q = 2$. This is needed for Theorem 2.3.

The *systems of roots* of Bourbaki [1] with 0 added are symmetrysets, as are the rootsystems defined below. Despite the apparent greater generality, we shall see that the rootsystems defined below are, up to isomorphism, the systems of roots of Bourbaki with 0 added. Thus, the classification in Bourbaki [1] applies to rootsystems.

2.2 Definition. A *rootsystem* is a finite productset R such that for all $a \in R$, $a \neq 1$, there exists $a^{\cdot} \in \mathrm{Hom}(R, \mathbb{Z})$ such that $a^{\cdot}(a) = 2$ and $s_a(b) = a^{-a^{\cdot}(b)}b$ defines an automorphism and symmetry s_a of R at a.

2.3 Theorem. *The following conditions are equivalent*:

(1) *R is a rootsystem*;

(2) *R is a symmetryset and* $\mathrm{Hom}(R, \mathbb{Z})$ *separates* R;

(3) *R is a reflectionset and* $a^* \in \mathrm{Hom}(R, \mathbb{Z})$ *for all* $a \in R$, $a \neq 1$;

(4) *R is isomorphic to a system of roots in the sense of Bourbaki* [1] *with* 0 *added.*

Proof. We show that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$. Suppose first that R is a rootsystem, $b, c \in R$, $b \neq c$ and $a^{\cdot}(b) = a^{\cdot}(c)$ for all $a \in R$, $a \neq 1$. Consider first the possibility that $c^{-1}b \in R - \{1\}$. Then $c \in R$, $c^{-1}b \in R$ and $c(c^{-1}b) =$

$b \in R$ implies that $a^{\cdot}(b) = a^{\cdot}(c(c^{-1}b)) = a^{\cdot}(c) + a^{\cdot}(c^{-1}b)$, so that $a^{\cdot}(c^{-1}b) = a^{\cdot}(b) - a^{\cdot}(c) = 0$ for all $a \in R$, $a \neq 1$. In particular, $2 = (c^{-1}b)^{\cdot}(c^{-1}b) = 0$, a contradiction. Thus, $c^{-1}b \notin R - \{1\}$ so that $R_b(c) = \{b, ..., c^q b\}$ and $R_b(c)$ is bounded of length q. It follows that $s_c(b) = c^q b$, so that $s_c(b) = c^{-c^{\cdot}(b)}b$ implies that $-q = c^{\cdot}(b) = c^{\cdot}(c) = 2$, a contradiction. We must conclude that $\{a^{\cdot} \mid a \in R, a \neq 1\}$ separates R, so that $(1) \Rightarrow (2)$. For $(2) \Rightarrow (3)$, suppose that R is a symmetryset and $\mathrm{Hom}(R, \mathbb{Z})$ separates R. Let R^* be tha additive group $\mathrm{Hom}(R, \mathbb{Z})$, let R^{**} be the additive group $\mathrm{Hom}(R^*, \mathbb{Z})$ and define $\hat{a} \in R^{**}$ for $a \in R$ by $\hat{a}(f) = f(a)$ $(f \in R^*)$:

$$\hat{a}(f+g) = (f+g)(a) = f(a) + g(a) = \hat{a}(f) + \hat{a}(g).$$

Then $\wedge: a \mapsto \hat{a}$ is a homomorphism from R to $\hat{R} = \{\hat{a} \mid a \in R\}$ which is bijective, since $\mathrm{Hom}(R, \mathbb{Z})$ separates R. Since $\hat{R}$ is contained in the torsion free group R^{**}, the strings $\hat{R}_{\hat{b}}(\hat{a})$ are bounded for all $\hat{a}$, $\hat{b} \in \hat{R}$, $\hat{a} \neq \hat{1}$. It follows that the strings $R_b(a)$ are bounded for all $a, b \in R$, $a \neq 1$, so that the symmetries s_a are the reflections r_a and R is a reflectionset. Finally, for b, c, $bc \in R$ we have $a^{-a^*(bc)}bc = r_a(bc) = r_a(b)\ r_a(c) = (a^{-a^*(b)}b)(a^{-a^*(c)}c)$, so that $\hat{b} + \hat{c} - a^*(bc)\,\hat{a} = \hat{b} - a^*(b)\,\hat{a} + \hat{c} - a^*(c)\hat{a}$, and $a^*(b) = a^*(b) + a^*(c)$ since the additive group R^{**} is torsion free. Thus $a^* \in \mathrm{Hom}(R, \mathbb{Z})$ for all $a \in R$, $a \neq 1$. Clearly, $(4) \Rightarrow (1)$. It remains to show that $(3) \Rightarrow (4)$. For this, assume that R is a reflectionset and $a^* \in \mathrm{Hom}(R, \mathbb{Z})$ for all $a \in R$, $a \neq 1$. As in $(1) \Rightarrow (2)$, $\{a^* \mid a \in R,\ a \neq 1\}$ separates R. Therefore, as in $(2) \Rightarrow (3)$, $R \to \hat{R}$ is a bijective homomorphism with $\hat{R}$ abelian. But this homomorphism is an isomorphism by Theorem 2.4 below, and $\hat{R}$ is isomorphic to a system of roots with 0 added by the discussion below. ▮

The strategy in the above proof is to pass from a symmetryset R to its image R under the *closure homomorphism* $\wedge: R \to R^{**}$ where $\hat{a}$ is defined by $\hat{a}(f)(f \in R^*)$ for $a \in R$. Here, $R^* = \mathrm{Hom}(R, \mathbb{Z})$ and $R^{**} = \mathrm{Hom}(R^*, \mathbb{Z})$ are torsion free additive groups. Since automorphisms s of R determine adjoints $s^* \in \mathrm{Aut}\, R^*$ and $s^{**} = (s^*)^* \in \mathrm{Aut}\, R^{**}$ such that $s^{**}(\hat{a}) = \widehat{s(a)}$, the image $\hat{R}$ of the symmetryset R is a reflectionset called the *closure* of R. Here s^* is defined, as one would expect, by $s^*(f) = f \circ s$ for $f \in R^*$. More specifically, a symmetry $s_a \in \mathrm{Aut}\, R$ of R at $a \neq 1$ determines a symmetry $\hat{s}_a = (s_a)^{**}|_{\hat{R}} \in \mathrm{Aut}\, \hat{R}$ of $\hat{R}$ at $\hat{a}$ which, since R^{**} is torsion free, is the reflection of R^* at $\hat{a}$: $\hat{s}_a = r_{\hat{a}}$. As in the above proof, we have $\hat{a}^* \in \mathrm{Hom}(\hat{R}, \mathbb{Z})$ for all $\hat{a} \in \hat{R}$, $\hat{a} \neq 0$. Since $r_{\hat{a}}(\hat{b}) = \hat{b} - \hat{a}^*(\hat{b})\hat{a}$ $(\hat{a} \in \hat{R}, \hat{a} \neq 0)$, $\hat{R}$ is isomorphic to a system of roots in the sense of Bourbaki [1] with 0 added, the system of roots being the subset $(\hat{R} - \{0\}) \otimes 1$ of the subspace V which it generates in $R^{**} \otimes_{\mathbb{Z}} \mathbb{R}$.

The closure homomorphism $R \to \hat{R}$ is an isomorphism if and only if it is injective, that is, if and only if $\mathrm{Hom}(R, \mathbb{Z})$ separates points, by the following theorem.

2.4 THEOREM. *Let* $s: R \to R'$ *be a surjective homomorphism of productsets, denoted* $s(b) = b'$. *Suppose that all strings* $R_b(a)$ $(a, b \in R, a \neq 1)$ *and* $R'_{b'}(a')$ $(a', b' \in R', a' \neq 1')$ *are bounded and* $r_{a'}(b') = r_a(b')$ *for all* $a, b \in R$, $a \neq 1$, $a' \neq 1'$. *Then*

(1) $a'^*(b) = a^*(b)$ *for all* $a, b \in R$, $a \neq 1$, $a' \neq 1'$;

(2) *for any* $a \in R$, $c' \in R'$, $a' \neq 1'$, *there exists* $b \in R$ *such that* $b' = c'$ *and* s *maps* $R_b(a)$ *bijectively to* $R_{b'}(a')$;

(3) s *is an isomorphism if and only if* s *is bijective.*

Proof. For (1), we have $r_{a'}(b') = r_a(b)' = (a^{-a^*(b)}b)' = a'^{-a^*(b)}b'$ and $r_{a'}(b') = a'^{-a'^*(b')}b'$. Since s is a homomorphism, we have $-r' \leqslant -r \leqslant -a^*(b) \leqslant q \leqslant q'$, where $R_b(a) = \{a^{-r}b, ..., a^q b\}$ and $R_{b'}(a') = \{a'^{-r'}b', ..., a'^{q'}b'\}$. This implies that $a^*(b) = a'^*(b')$, proving (1). For (2), let $R'_{c'}(a') = \{a'^{-r'}c', ..., a'^{q'}c'\} = \{c'_0, ..., c'_{n'}\}$ with $n' = q' + r'$ and $c'_i = a'^i c'_0$ for $0 \leqslant i \leqslant n'$. Let $R_{c_0}(a) = \{b_0, ..., b_n\}$, where $b_j = a^j b_0$ $(0 \leqslant j \leqslant n)$, and note that $c_0 = b_0$, since $c_0 = a^j b_0$ for some j and $a'^{-1}c'_0 \notin R'$. Then $c'_{n'} = r_{a'}(c'_0) = r_a(c_0)' = r_a(b_0)' = b'_n = (a^n b_0)' = a'^n b'_0 = a'^n c'_0 = c'_n$. Thus, $n' = n$ and $b'_j = (a^j b_0)' = a'^j c'_0 = c'_j$ for $0 \leqslant j \leqslant n$, so that s maps $R_{c_0}(a) = \{b_0, ..., b_n\}$ bijectively to $\{c'_0, ..., c'_n\} = R'_{c'}(a')$. Clearly, $R_{c_0}(a) = R_b(a)$, where $b = b_i$, and i is chosen with $1 \leqslant i \leqslant n$ so that $b'_i = c'$ and $b' = c'$. For (3), let s be bijective and a, $b \in R$, $a \neq 1$ such that $a'b' \in R'$. With notation as in (2) above, we then have $R_b(a)' = \{b_0, ..., b_n\}' = \{b'_0, ..., b'_n\} = R_{b'}(a')$, $b = b_r = a^r b_0$ for some r, $0 \leqslant r \leqslant n$, $b' = (a^r b_0)' = a'^r b'_0$ and $a'b' = a'^{r+1}b'_0 = b'_{r+1} = (a^{r+1}b_0)' = (aa^r b_0)' = (ab)'$.

Thus, $ab \in R$ and $s^{-1}(a'b') = ab = s^{-1}(a')\, s^{-1}(b')$. This shows that $s^{-1} \in \operatorname{Hom}(R', R)$, so that s is an isomorphism. ∎

3. DIRECT SUMS

Let $R_1, ..., R_n$ be productsets. Resituate them (up to isomorphism) so that their identities all coincide (call it 1) and $R_i \cap R_j = \{1\}$ for all $i \neq j$. Then the *outer direct sum* $R = R_1 \oplus \cdots \oplus R_n$ of $R_1, ..., R_n$ is the productset $R = R_1 \cup \cdots \cup R_n$ such that $ab = c$ in R if and only if there exists i such that $a, b, c \in R_i$ and $ab = c$ in R_i.

Conversely, for any productset R with identity 1, R is the *inner direct sum* $R = R_1 + \cdots + R_n$ of subsets $R_1, ..., R_n$ of R if $R = R_1 \cup \cdots \cup R_n$, $R_i \cap R_j = \{1\}$ for all $i \neq j$ and $ab = c$ in R if and only if there exists i such that $a, b, c \in R_i$ and $ab = c$ in R_i. Note that if R is an inner direct sum $R = R_1 + \cdots + R_n$, then

(1) $R_1, ..., R_n$ are productsets whose outer direct sum is R;

(2) $R_b(a) = R_{jb}(a)$ for $a \in R$, $b \in R_j$, and $R_b(a) = \{b\}$ if $b \in R_j$ and $a \in R - R_j$;

(3) R is a symmetryset (respectively reflectionset) if and only if the $R_1, \ldots, R_n$ are also;

(4) a symmetry (respectively reflection) s_a of R at $a \in R_i$ is the same as a symmetry (respectively reflection) s_{ia} of R_i at a extended to R by fixing the elements of $R - R_i$.

Let R be a symmetryset and let S, $T \subset R$. We let $ST = \{ab \mid a \in S, b \in T, ab \in R\}$, $S^{-1} = \{a^{-1} \mid a \in S\}$ and $\dot{S} = S - \{1\}$. We say that S is *π-closed* if $SS \subset S$ and *symmetric* if $S^{-1} \subset S$.

A subset $\dot{S}$ of $\dot{R}$ is *closed* in $\dot{R}$ if $R = S + \tilde{S}$ (inner direct sum), where $S = \dot{S} \cup \{1\}$ and $\tilde{S} = R - \dot{S}$. For $\dot{S}$ closed, S and $\tilde{S}$ are clearly π-closed and symmetric, and they can be viewed as symmetrysets. Moreover, $\tilde{S}$ is closed and $\tilde{\tilde{S}} = S$. For $\dot{S}$ and $\dot{T}$ closed in R, the inner direct sum $R = (S \cup \tilde{S}) \cap (T \cup \tilde{T}) = S \cap T + S \cap \tilde{T} + \tilde{S} \cap T + \tilde{S} \cap \tilde{T}$ can be used to show that $\dot{S} \cap \dot{T}$ and $\dot{S} \cup \dot{T}$ are closed in $\dot{R}$. Thus, the set $\mathscr{S} = \{\dot{S} \mid \dot{S}$ is closed in $\dot{R}\}$ is a topology for $\dot{R}$. Since $\dot{S}$ is closed if and only if $\dot{S}$ is open, the connected components of $\dot{R}$ are the minimal elements $\dot{S}_1, \ldots, \dot{S}_n$ of $\mathscr{S}$ and $R = S_1 + \cdots + S_n$ *is the unique inner direct sum decomposition of R which cannot further be refined.* If $n = 1$, we say that R is *irreducible*. The irreducible symmetrysets S_i are the *components* of R.

If R is a subset of a group, then the topology $\mathscr{S}$ introduced here for $\dot{R}$ coincides with the symmetric G-topology defined in Winter [8] for any finite subset $\dot{R}$ of a group G by the condition $S \subset \dot{R}$ is *closed* if S and $\dot{R} - S$ are both π-closed and symmetric.

Examples of symmetrysets which are not root systems are symmetrysets having a component which is a nonzero finite vectorspace $V = (\mathbb{Z}_p)^d$, a symmetry $s_a(v)$ at a being defined for any $\dot{a} \in \operatorname{Hom}(V, \mathbb{Z}_p)$ such that $\dot{a}(a) = \bar{2}$ by $s_a(v) = v - \dot{a}(v)a$ $(a \in V,\ a \neq 0)$.

Any two such symmetries s_{1a}, s_{2a} are conjugate in Aut V by a *unipotent* $u_a \in$ Aut V of V at a: $u_a(a+b) = a + u_a(b)$ for all $b \in V (p > 2)$. In fact, it is the conjugacy of 2-Sylow groups in the dihedral group $\langle s_{1a}, s_{2a} \rangle$. This unicity can be generalized along the lines of Winter [7].

4. Reduced Symmetrysets in an Abelian Group

Let R be a subset containing 0 of an additive abelian group G having no 2, 3, 5, 7 torsion, and assume that R is a symmetryset which is *reduced*, that is, $2a \notin R$ for all $a \in R$, $a \neq 0$.

Using simple modifications of techniques of Seligman [5], we show that R is a rootsystem. We first note that a string $R_b(a)$ cannot contain b, $b + a$,

$b + 2a$, $b + 3a$, $b + 4a$. For suppose otherwise. Since $\pm 2a \notin R$, the elements b, $b + a$, $b + 2a$, $b + 3a$, $b + 4a$ are nonzero. Since $2(b + a) \notin R$ and $-2a \notin R$, $R_b(b + 2a) = \{b\}$, so that $s_{b+2a}(b) = b$. Since $2(b + 3a) \notin R$ and $2a \notin R$, $R_{b+4a}(b + 2a) = \{b + 4a\}$ so that $s_{b+2a}(b + 4a) = b + 4a$. Letting $s = s_{b+2a}$, we then have $b + 4a = s(b + 4a) = s(b) + 4s(a) = b + 4s(a)$. Thus, $4a = 4s(a)$ and $a = s(a)$. But then $-(b + 2a) = s_{b+2a}(b + 2a) = s(b + 2a) = b + 2a$, so that $2(b + 2a) = 0$. Thus, $b + 2a = 0$ and $-2a \notin R$, a contradiction.

It follows that the strings $R_b(a)$ are bounded of length at most 3. Therefore, the symmetries $s_a(b)$ are the reflections $r_a(b) = b - a^*(b)a$, $a^*(b) = r - q$. Since $r_a(b + c) = r_a(b) + r_a(c)$, we have $a^*(b + c)a = (a^*(b) + a^*(c))a$ for all $b, c, b + c \in R$. Since G has no 2, 3, 5, 7 torsion, the order of a is at least 11, so that the restrictions on the Cartan integers $r - q$ that $r, q \geqslant 0$ and $r + q \leqslant 3$ lead us to conclude that $a^*(b + c) = a^*(b) + a^*(c)$. It follows that $a^* \in \mathrm{Hom}(R, \mathbb{Z})$ for all $a \in R$, $a \neq 0$, so that R is a rootsystem.

4.1 Theorem. *R is isomorphic to a reduced system of roots in the sense of Bourbaki* [1] *with* 0 *added.*

Proof. This now follows from Theorem 2.3 since R is a rootsystem. ∎

5. Classical Lie Algebras

Let L be a classical Lie algebra with classical Cartan subalgebra H and corresponding set of roots $R = R(L, H)$. Suppose that the characteristic of L is not 2, 3, 5, 7. Then $R = R(L, H)$ is a reduced reflectionset with reflections $r_a(b) = b - 2(b(h_a)/a(h_a))a$, by the beautiful results of Lemmas II2.2, II3.1, II3.2, II4.2 of Seligman [5]. Thus, $R = R(L, H)$ is isomorphic to a reduced system of roots in the sense of Bourbaki [1] with 0 added, by Theorem 4.1. Since L together with H is determined uniquely up to isomorphism by $R = R(L, H)$, by a version of Theorem 3.7.4.9. of Winter [6], the classical Lie algebras L with H are classified by systems of roots in the sense of Bourbaki [1] with 0 added by the correspondence $(L, H) \mapsto R(L, H)$. In this approach to the classification, we transfer R together with its combinatorial structure from the hostile environment of characteristic p to real Euclidean space by passing to $\hat{R} \subset R^{**}$, rather than classify R "on the spot" using orderings, fundamental systems of roots and their Cartan matrices.

For $p = 2, 3, 5, 7$, see Brown [2, 3] and Mills [4].

Relationsets and Symmetrysets

A *relationset* is a set R and a subset π of R^3, such that R has an *identity element* 1 such that $(x, 1, x)$, $(1, x, x) \in R$ for all $x \in R$. We let $ax = y$ indicate that $(a, x, y) \in \pi$. Note that y need not be uniquely determined by a and x.

Clearly, a productset is just a relationset such that

(1) $ax = y$ and $ax = z$ implies $y = z$; and

(2) $ay = x$ and $az = x$ implies $y = z$ for all a, x, y, $z \in R$.

In this section, we let R be a finite relationset. Our objective is to:

(1) indicate briefly how the above theory of symmetries generalizes from productsets to relationsets;

(2) describe the rootsystem $\widehat{S(R)}$ which R determines.

A *homomorphism* of relationsets R_1, R_2 is a mapping $f: R_1 \to R_2$ such that $ax = y$ implies $f(a)f(x) = f(y)$ for all a, a, x, $y \in R$. And f is an *isomorphism* (*automorphism* if $R_1 = R_2$) if f is bijective and f, f^{-1} are homomorphisms. We let $\mathrm{Hom}(R_1, R_2)$ denote the set of homomorphism from R_1 to R_2 and we let $\mathit{Aut}\, R$ denote the automorphism group of R.

For $a \in R$ and $S \subset R$, a determines the equivalence relation on S generated by the relation $\{(x, y) \in S \times S \mid ax = y\}$. We let $S_b(a)$ denote the corresponding equivalence class of $b \in S$.

A *symmetry* of S at a is a bijection $s_a: S \to S$ such that

(1) $s_a S_b(a) = S_b(a)$ for all $b \in S$;

(2) $ax = y$ implies $as_a(y) = s_a(x)$ for all x, $y \in S$.

Note that if R is a productset, condition (2) is equivalent to $s_a(a^i x) = a^{-i} s_a(x)$ for all $a^i x \in R$.

A *symmetryset* is a finite relationset R such that $\mathrm{Aut}\, R$ contains a symmetry s_a of R at a for every $a \in R$. Clearly, this coincides with our earlier definition if R is a productset.

Consider the additive group $R^* = \mathrm{Hom}(R, \mathbb{Z})$, where $f + g = h$ is the function defined by $h(a) = f(a) + g(a)$ $(a \in R)$ for f, $g \in R^*$. Similarly, consider the additive group $R^{**} = \mathrm{Hom}(R^*, \mathbb{Z})$. As in Section 2, we get the *closure homomorphism* $\wedge: R \to R^{**}$ with $\hat{a}$ defined by $\hat{a}(f) = f(a)$ $(f \in R^*)$ for $a \in R$:

$$\begin{aligned} xy = z \text{ in } R &\Rightarrow f(x) + f(y) = f(z) && \forall f \in R^* \\ &\Rightarrow \hat{x}(f) + \hat{y}(f) = \hat{z}(f) && \forall f \in R^* \\ &\Rightarrow \hat{x} + \hat{y} = \hat{z} && \text{in } R^{**}. \end{aligned}$$

The subset $\hat{R} = \{\hat{a} \mid a \in R\}$ of the torsion free additive group R^{**} is a productset called the *closure* of R.

If R is a symmetryset in the sense of this section, then the productset $\hat{R}$ is a symmetryset since, as one easily checks, the double adjoint s_a^{**} of s_a restricts to a symmetry $\hat{s}_a = s_a^{**}|_{\hat{R}}$ of $\hat{R}$ at $\hat{a}$ in $\operatorname{Aut}\hat{R}$ In fact, $\hat{R}$ is a system of roots with 0 added, as we now show in greater generality.

For any relationset R, we let $S(R) = \{a \in R \mid \text{there exists a symmetry } s_a \in \operatorname{Aut} R \text{ of } R \text{ at } a\}$. Clearly, $S(R)$ is stable under $\operatorname{Aut} R$, so that $s_a S(R) = S(R)$ for all $a \in S(R)$. Thus, $S(R)$ is a symmetryset in R in the following sense.

6.1 Definition. A *symmetryset in R* is a subset S of R containing 1 such that for each $a \in S$, there exists a symmetry $s_a \in \operatorname{Aut} R$ of R at a such that $s_a(S) = S$. The *closure of S in R* is the image $\hat{S} = \{\hat{s} \mid s \in S\}$ of S under the closure mapping $\wedge: R \to \hat{R}$ of R.

For $a \in S(R)$, the double adjoint s_a^{**} is an automorphism of the torsion free group R^{**} and $\bar{s}_a = s_a^{**}|_{\langle R \rangle} \in \operatorname{Aut}\langle \hat{R} \rangle$ satisfies $\bar{s}_a(b) \equiv b \pmod{\mathbb{Z}\hat{a}}$ for $b \in \langle \hat{R} \rangle$. Thus, there exists $a^{\cdot} \in \operatorname{Hom}(\langle \hat{R} \rangle, \mathbb{Z})$ such that $\bar{s}_a(b) = b - a^{\cdot}(b)a$ $(b \in \langle \hat{R} \rangle)$. Clearly, $a^{\cdot}(\hat{a}) = 2$ since $-\hat{a} = \bar{s}_a(\hat{a})$.

6.2 Theorem. *$\widehat{S(R)}$ is a system of roots with 0 added.*

Proof. By the above discussion, $\widehat{S(R)}$ has the required reflections $r_{\hat{a}}(\hat{b}) = \hat{b} - a^{\cdot}(\hat{b})\hat{a}$ $(\hat{a} \in S(\hat{R}),\ \hat{a} \neq 0)$. ∎

In taking the closure $S(R)$ of $S(R)$ in R rather than in $\widehat{S(R)}$ in the above discussion, we assure the validity of the otherwise unwarranted step $\bar{s}_a(b) \equiv b \pmod{\mathbb{Z}\hat{a}}$ for b in the closure of $S(R)$ in $S(R)$. (The symmetry s_a might not preserve the sets $S(R)_b(a)$.)

6.3 Definition. $\widehat{S(R)}$ is the *rootsystem* of R.

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